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SATURATION AND TRAVELING WAVES

ROBI PESCHANSKI
Service de Physique Thorique, CEA/Saclay
URA 2306, unit de recherche associe au CNRS
91191 Gif-sur-Yvette cedex, France
E-mail: pesch@spht.saclay.cea.fr

High parton density effects with energy obey non-linear QCD evolution equations for which exact solutions are not known. The mathematical class to which the non-linear Balitsky-Kovchegov equation belongs is identified, proving the existence of asymptotic in energy traveling wave solutions which are "universal" i.e. independent of the initial conditions and of the precise form of the non-linearities. This has an direct impact on geometrical scaling and the diffusive transition to saturation, which is shown to be "normal" for constant QCD coupling and "abnormal" for running coupling.

1 Introduction

Considering the scattering of a hard projectile (e.g. a massive QCD dipole) on an extended target, the Balitsky Fadin Kuraev Lipatov (BFKL) [1] evolution equation implies a densification of gluons and sea quarks with incident energy, while they keep in average the same size. It is thus natural to expect [2] a modification of the evolution equation towards a *saturation* regime. Recently, a theoretical appoach to saturation has been found [3,4] related to non-linear evolution equations of the gluon density in the framework of perturbative QCD. In the transition to saturation, the exponential growth regime related to the BFKL kernel gets modified by non-linear terms, leading to the Balitsky-Kovchegov (BK) equation [4]. A more general non-linear functional equation is expected to take into account the multiple correlations and to describe the fully saturated phase [3]. The aim of our approach [5,6,7] is to explore the mathematical properties of the BK equation and derive its physical consequences for saturation in QCD.

2 Saturation and Non-Linear Equations

The Balitsky-Kovchegov (BK) equation [4] considers the energy evolution within the QCD dipole Hilbert space [8]. To be specific let us consider $N(Y, x_{01})$, the dipole forward scattering amplitude and define

$$\mathcal{N}(Y,k) = \int_0^\infty \frac{dx_{01}}{x_{01}} J_0(kx_{01})\, N(Y,x_{01}) \ . \tag{1}$$

Within suitable approximations (large N_c, summation of fan diagrams, spatial homogeneity), this quantity obeys (see the second reference in [4]) the nonlinear evolution equation

$$\partial_Y \mathcal{N} = \bar{\alpha}\chi\left(-\partial_L\right)\mathcal{N} - \bar{\alpha}\,\mathcal{N}^2 \ , \tag{2}$$

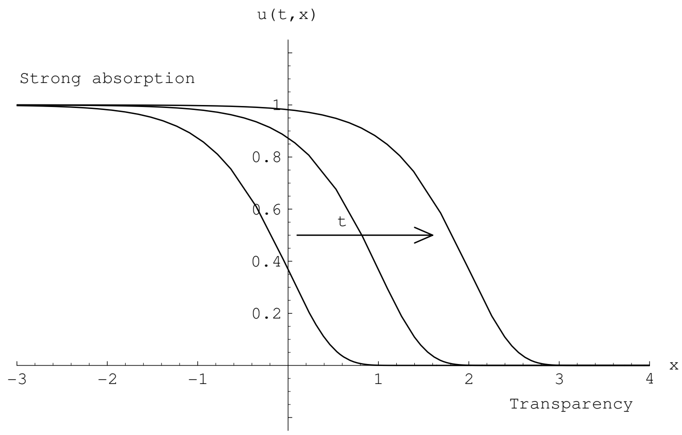

Figure 1. *Typical traveling wave solution.* The function $u(t,x)$ is represented for three different times. The wave front connecting the regions $u=1$ and $u=0$ travels from the left to the right as t increases. That illustrates how the "strong absorption" or saturated phase region invades the "transparency" region.

where $\bar{\alpha}=\alpha_s N_c/\pi$, $\chi(\gamma)=2\psi(1)-\psi(\gamma)-\psi(1-\gamma)$ is the characteristic function of the BFKL kernel [1], $L=\log(k^2)$. In a first stage [5], let us consider the kernel expanded up to second order around $\gamma=\frac{1}{2}$.

Eq. (2) boils down to a parabolic nonlinear partial derivative equation:

$$\partial_Y \mathcal{N}=\bar{\alpha}\left\{\chi\left(\tfrac{1}{2}\right)+\tfrac{1}{2}\chi''\left(\tfrac{1}{2}\right)\left(\partial_L+\tfrac{1}{2}\right)^2\right\}\mathcal{N}-\bar{\alpha}\mathcal{N}^2\ . \tag{3}$$

The mathematical point [5] of our recent approach is to remark that the structure of Eq.(3) is identical (by a suitable linear redefinition $\mathcal{N}(L,Y)\to u(x,t)$ and for fixed α) to the Fisher and Kolmogorov-Petrovsky-Piscounov (F-KPP) equation [9]:

$$\partial_t u(t,x)=\partial_x^2 u(t,x)+u(t,x)(1-u(t,x)) \tag{4}$$

which appeared in the problem of gene diffusion and annihilation (1938).

3 Universality and Traveling Wave Solutions

The remarkable mathematical property of the F-KPP equation is the existence of traveling wave solutions of the F-KPP equation [9] at large times. This means that there exists a function of one variable w such that

$$u(t\to+\infty,x)\sim w\left\{x-2t-\tfrac{3}{2}\log t+\mathcal{O}(1)\right\} \tag{5}$$

uniformly in x. Such a solution is depicted on Fig.(1).

This analysis can be extended [6] to the study of the equation with the full kernel. Indeed, only the second-order expansion around a given critical value $\gamma=\gamma_c=.6275...$ is relevant. Let us describe its general consequences. The well-known

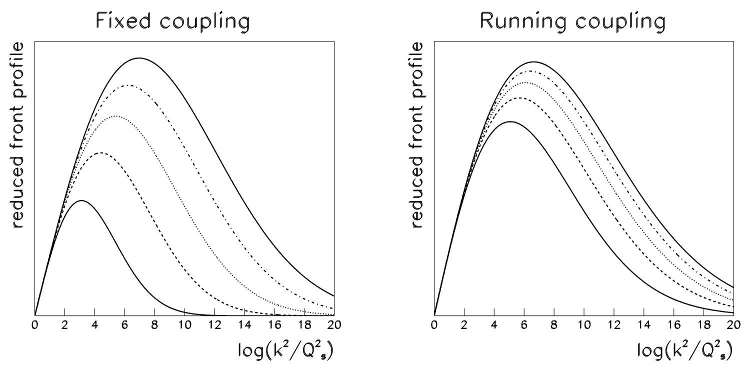

Figure 2. *Evolution of the reduced front profile.* Fixed coupling: left; Running coupling: right. The reduced front profile $(k^2/Q_s^2)^{\gamma_c}\,\mathcal{N}(k/Q_s(Y),Y)$ is plotted against $\log(k^2/Q_s^2)$ for different rapidities. The various lines correspond to rapidities from 2 (lower curves, full line) up to 10 (upper curves). Note the similarity of the wave fronts, but the quicker time evolution (in $\sqrt{t}$) for fixed coupling, by contrast with the slow time evolution (in $t^{1/3}$) for the running coupling case.

geometric scaling property [10] is obtained for the solution of the non-linear equation (3) at large enough energy. In our notation, the geometric scaling property can be written

$$\mathcal{N}(Y,x_{01}) = \mathcal{N}\left(x_{01}Q_s(Y)\right)\ , \tag{6}$$

where

$$Q_s^2(Y) = \exp\left\{\bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c}Y - \frac{3}{2\gamma_c}\log Y - \frac{3}{(\gamma_c)^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\frac{1}{\sqrt{Y}} + \mathcal{O}(1/Y)\right\}\ , \tag{7}$$

plays the role of the saturation scale squared. Note that the solution (5) mathematically requires an value initial condition $\mathcal{N}_0(k{\rightarrow}{+\infty},Y_0) \ll 1/k^{2\gamma_c}$ which is realized in first order QCD by color transparency $\mathcal{N}_0 \sim 1/k^2$. Note that the result gives a rigorous proof of previous evaluations based on linear evolution with boundary conditions (first term: [11], second term: [12]); the third is new [7].

It is possible to show that the result (5) is more general, by various extensions of the F-KPP solutions. First, some general arguments confirmed by numerical simulations (see the review in [9]) lead to expect the same result for the full nonlinear equation (2). It is independent of the precise form of the non-linear damping terms and from the initial conditions (provided the transparency condition is fulfilled). Hence the "Universality" property. Second, the results can be extended to running α. One interesting difference [6] with the fixed α case is the "abnormal" diffusion approach to scaling (in $t^{1/3} \sim Y^{1/6}$ instead of $t^{1/2} \sim Y^{1/2}$), see Fig.(2).

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